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ENTERING THE WORLD OF MATHEMATICS RESEARCH AT SCHOOL AGE

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The talent is a resource which, unlike the ores, could vanish if not discovered early enough ... P. Kenderov

The paper deals with gifted education programs for enhancing the potential and the skills of highly achieving high school students for doing research in mathematics. The authors share their experience and impressions from their involvement in national and international context.

1. Introduction. In recent years, many studies have highlighted an alarming decline in young people's interest for science studies and mathematics. This lack of interest is an issue of great importance addressed by recent reports of the European Commission [1] and European projects (cf. [2]). There is firm evidence that indicates a connection between this phenomenon and the way students are taught.With so many reforms in mathematical education worldwide, we would still agree with G. H. Hardy that *if the fine arts were taught in the same way, they would be reduced to studying techniques for clipping stone and mixing paints.*

One of the reasons many students come to view mathematics as a static body of facts and knowledge divorced from their world of experience is that less than 1% of mathematics concept they learn were discovered after the 18^{th} century [3]. Establishing connections between the students' study of mathematics and the work of the current mathematicians can *breathe life* into the mathematics classroom. Not only should the students be given the chance of seeing the real nature of mathematics at school age but they can begin *doing mathematics* in their schooldays. There is no doubt that *being a mathematician is no more definable as 'knowing' a set of mathematical facts than being a poet is definable as knowing a set of linguistic facts* [4]. Some modern reformers of the mathematics education would comment as follows: "Yes, they must understand, not merely know." But this misses the capital point that *being a mathematician, again like being a poet, or a composer or an engineer, means doing, rather than knowing or understanding.*

In this paper we concentrate on some approaches of working with high school students in the frames of specially designed research programs and we share our experience in national and international context. We pay special attention to the role of working on research projects in enhancing students' math competence and motivation to study mathematics or related to it fields.

2. Educational programs outside of schools. To appreciate the real beauty and meaning of mathematics as a scientific field and possibly choose it as their future profession the students should be enabled to participate in some forms in which: to use mathematics in daily life activities; to apply mathematical thinking and modelling so as to solve problems that arise in other fields; to use mathematical methods as an integrated whole; to formulate their own hypotheses and problems, and to attack open problems.

For students at school age to experience at least partially these sides of the math research process various forms exist such as: specialized research programs, school sections in the frames of professional conferences, symposia and fairs for young scientists.

Many researchers in gifted education believe that educational programs outside of schools are absolutely necessary for gifted children because they meet their special learning needs by providing more opportunities for independent inquiry, in-depth study, and accelerated learning [5]. In addition, a *summer program is a great chance to meet other bright kids who are fascinated by learning. These are their true intellectual peers. Courses in these programs combine the best of both worlds: accelerated content and bright age-peers* [6]. Summer programs vary in terms of content, duration, intensity, sponsorship, and overall purpose. Still some general benefits are found to include the following:

- Perceptions of increased social support for learning and achievements due to homogeneous grouping and support from counselors, tutors, and mentors;
- Positive feeling resulting from a more appropriate match between the student's academic potential and the challenge of the courses and the research projects;
- Development of intensive-study skills and skills for carrying out scientific research;
- Reinforcement for risk taking as a result of extending oneself intellectually and socially;
- Growth in acceptance of others and (in the case of international component) knowledge of different cultures.

Below we shall describe the infrastructure of two research programs in which the authors have been involved – the *Research Science Institute* (RSI) held at the Massachusetts Institute of Technology (MIT, US), and the Bulgarian *High School Students Institute of Mathematics and Informatics* (HSSI).

3. Good practices in world context – the RSI summer program. The RSI was developed by the Center of Excellence in Education, a non-profit educational foundation in McLean, Virginia. The Center was founded by the late Admiral Rickover and Joann DiGennaro in 1983, with the express purpose of nurturing young scholars to careers of excellence and leadership in science, mathematics, and technology. Central to CEE is the principle that talent in science and math fulfills its promise when it is nurtured from an early age**.** RSI is an intensive annual six-week summer program which is sponsored jointly by CEE and MIT. It is attended by approximately 80 high-school students from US and other nations including Bulgaria, China, France, Germany, Greece, Hungary, Israel, Lebanon, Poland, Singapore, Saudi Arabia, and the UK. Once selected, the students go to MIT and work on a research project under the guidance of faculty, post-docs, and graduate students at MIT, Harvard, Boston University, and other research institutions from Boston-area. All the students chosen for the RSI have already acquired a deep interest in a scientific field– mathematics, CS and natural sciences. The Institute begins with four days of formal classes. Professors of mathematics, biology, chemistry and physics give lectures on important aspects of their field and their own research. The students also attend evening lectures in science, philosophy, and humanities delivered by eminent researchers including Nobel Prize winners. The internships that follow the first week classes comprise the main component of the RSI. Students work in their mentors' research laboratories for 5 weeks. At the end of the internship they present a paper on their research and give an oral presentation in front of a large audience at the RSI Symposium.

In order that this be an orderly and seamless intellectual process, it is best to characterize the RSI research paper as a *progress report for a continuing research effort.* As expressed by Dr. John Dell, the RSI Director in 2001 – 2002, *it is more useful to think of the RSI paper in this way than as a paper about a finished research project because this model allows students to write progressive versions of the paper and to prepare presentations of their work throughout the program using a consistent intellectual template to which the tutoring staff can target their support.* Progress reports typically focus more on methods and process than a final research paper but they naturally evolve into final reports as some original results are obtained. The transition from progress report to final research paper is typically one of reduction through editing of existing text with the perspective of the final results in mind. RSI is well structured for this reduction process as last week teaching assistants and *nobodies* (RSI alumni with no formal duties) supply great quantities of quality editing advice in the week before the papers are due.

Especially important in the process of preparation are the *milestones* – intermediate steps of the process. Typical milestones for the written presentation are: *writing about a mini-project using the same sample as the one for the final paper; gradually filling the proposed sample starting with the background of the project, the literature studied and the methods used; considering partial cases and possible generalizations; classifying the cases of failure*, etc. Possible milestones for the oral presentation are: *speaking for 3 min on a freely chosen topic*, *presenting the introductory part of the project for 5 min*, etc.

All the milestones are accompanied by a feedback from the tutors who work closely with the students - they read and critique the draft papers, provide editorial remarks, suggest avenues of research and areas of additional background reading, give ideas for improvement of the oral presentations, etc. In general, tutors are the *psychological oil* if the students experience problems and lack of self-confidence.

To get an idea of the variety of topics of projects performed at RSI you might look at the compendiums of three consecutive years [7] containing the abstracts of all the written reports with five selected as representative which are published in full.

4. The High School Students Institute of Mathematics and Informatics. The long-term collaboration between the Center for Excellence in Education, Institute of Mathematics and Informatics at the Bulgarian Academy of Sciences (IMI-BAS) and *St. Cyril and St. Methodius* Foundation was one of the crucial factors for the establishment of the *High School Students Institute of Mathematics and Informatics* (HSSI) in the World Mathematical Year 2000 [8]. Founders of HSSI are the Union of Bulgarian Mathematicians (UBM), the Foundation "Evrika", the International Foundation "St Cyril and St Methodius" and IMI-BAS. One of the main ideas behind HSSI was *to implement RSI-like activities in Bulgaria, taking into account the local conditions and traditions.* This institute inherited the good traditions of an earlier movement of the technically creative youth in Bulgaria [9]. The local conditions included the infrastructure and activities of the UBM which has long-standing traditions in early identification and proper enhancement of talents. Since 1980, School sections in the framework of the annual Spring Conferences of UBM have been organized. This contributed naturally to the mission of HSSI *to keep the traditions alive giving them new spirit and new content.* Another important component of the local conditions is the environment provided by the IMI–BAS stimulating the growth and the progress of HSSI – library, Internet, rooms and equipment. Many researchers at the Institute devote significant part of their free time to keep the level of extra-curricular work with talented students high [Appendix, I]. Their work supports and enables *HSSI to assist the intellectual and professional growth of the high school students.*

The participants in HSSI are high school students between 8^{th} and 12^{th} grade, mainly from specialized science and mathematics secondary schools. Every participant in HSSI works individually (or in a team) on a freely chosen topic in mathematics, informatics and/or Information Technologies (IT) under the guidance of a teacher or another specialist. A written presentation of the project is sent to HSSI. All papers are refereed by specialists and the reviews are given to the authors. Papers involving creativity elements are given special credit. The best projects are accepted for presentation in the conference sessions of HSSI. A general view of the topics of the mathematics projects developed in HSSI is given in the Appendix, II.

HSSI organizes three events per year – two conference sessions and a research summer school (RSS). The *High School Students' Conference* is held in January at the Plovdiv branch of the HSSI (the Faculty of Mathematics and Informatics at Plovdiv University) and is usually attended by more than 200 students, teachers, researchers in mathematics and informatics, parents, journalists. The conference is held in two streams – mathematics and informatics/IT. The authors present their work in front of a Jury of specialists in the field and in the presence of a general audience. The jury can ask the students various questions so as to check the level of their understanding and creativity. Parallel to these two streams is a poster session. Based on the merits of the paper and the style of presentation, the Jury selects the best ones. Their authors receive *Certificates for excellence* and are invited for an interview for selecting two Bulgarian participants in RSI [Appendix, IV] and to participate in the *School Section* of the Spring Conference of UBM. The School Section is an independent event – it could be attended by students who present their research for a first time. The process of reviewing and selecting papers for the *School Section* is the same as above. The authors of the best projects from this section are invited to participate in the Research Summer School.

One of the most important components of HSSI is the three-week *Research Summer School*. The RSS takes place in July–August in Varna and Usana. During the first two weeks, lectures and seminars in mathematics are delivered by eminent specialists from universities, academic institutions and software companies. The main goal of the training is to extend the students' knowledge in topics related to their interests and to offer new problems to be studied and solved in further projects. Since 2005, the emphasis is on the development of *short-term project* under the guidance of mentors [Appendix, III]. The third week is devoted to a *Students' workshop*, where the participants report on their results and share ideas for further studies.

To help teachers improve their mentoring skills a *Teachers workshop* is also organized during the third week of the RSS. The participants are the mentors of the students' projects, presented at the events of HSSI during the school year.

Another important activity of the HSSI is its monthly seminar at the Institute of Mathematics and Informatics. The aim of the Seminar is to bring together high school students, teachers and scientists to present and discuss problems of common interest.

The 9-year activities of the HSSI have been recognized in a frames of the European projects *Meeting in Mathematics* and *Math2Earth*, and in the project proposal (*The Scientist in the Learner*).

5. A math project – guided research and presentation. Working on projects develops naturally skills important in life of the professional mathematicians – planning, searching for and selecting appropriate information, integrating knowledge from different fields (including informatics and IT), working in a team. Our experience shows that the activities could be successfully grouped in several phases:

- Preparation phase – motivating the students for exploring a topic of interest by delivering short lectures and appropriate warm-up problems
- Research phase – engaging the students in research activities by formulating appropriate:
  - short-term projects (expected to be developed for at most two weeks during a summer school or during the school year);
  - long-term projects (lasting from 6 weeks to 4-5 months, in some cases – up to 2 years);
- Extending the project with the formulation of open problems;
- Presentation phase – building up ICT enhanced skills for a written and oral presentation of the project [10];
- Passing on *the torch* – teaching students to act like mentors.

Let us illustrate some essential aspects of the students' math research and presentation.

Example 1: **A real project is open-ended**

One of the most important goals of HSSI is to enable the students with special interests in mathematics to experience the spirit of the research process. This process differs significantly from the problem solving at mathematics Olympiads since it doesn't end with finding the answer of a specific problem but often with posing new problems. Here is an example.

The first author: *Twenty years ago I came across the following problem formulated by Malfatti in 1803:* Cut three circles from a given triangle so that the sum of their areas is maximal. *In spite of the great interest this seemingly easy problem had raised, it was solved two centuries later by V. Zalgaller and G. Loss* [11]. *In conversations with students and teachers I have shared my opinion that the Malfatti problem is very suitable for a student research project since it provides rich opportunities for explorations of different variations of it. Three years ago, Emil Kostadinov (a student of the Mathematics High school in Blagoevgrad) presented a project on this topic* [2, p. 165] *which attracted my attention mostly with the authors' comments in the concluding part. Here they are* in a nutshell. *It follows from the theorem of Zalgaller and Loss that the solution of the Malfatti problem could be obtained by using the so-called* greedy algorithm, *viz. at each step we cut the largest possible circle. In Emil's project it is proved that the same is also true for the Malfatti problem in the case of square. Of course, one can formulate various Malfatti type problems and it is tempting to conjecture that their solutions could also be obtained by using the greedy algorithm. However Emil shows that this is not true in general. To do this he considers the following analog of the Malfatti problem:* Cut three nonintersecting triangles from a given circle so that the sum of their areas is maximal. *In this case the greedy algorithm does not lead to a solution as it can be derived by using the well-known fact that among all the n-gons inscribed in a circle the regular ones have* *maximal area. So it is natural to conjecture that the solution of the latter problem is given by three triangles forming a regular pentagon, inscribed in the given circle. To the best of my knowledge Emil's conjecture is still an open problem and I think it is a nice challenge to the students who like combinatorial geometry.*

Example 2: **From an IMO problem to a high-level theory**

The second author (HSSI 2003): *During the HSSI Summer school we were introduced to the problems of the most recent for the time IMO. The most interesting for me was the last one – to prove that a specific Diophantine power congruence doesn't have solution modulo some prime number. The solution of the problem required a standard trick but what attracted my attention was a comment given by a specialist in number theory during the Olympiad. As it turned out not only does there exist such a prime, but there are infinitely many of them for which the given congruence doesn't have a solution. Moreover, we can compute how infinitely many these primes are, viz. we can find their density! All this is provided by an outstanding result in algebraic number theory due to the Russian mathematician Nikolai Chebotarev* (*known in the literature as the Chebotarev Density Theorem*).

It was for quite a long period that my mind was occupied by the above comment – during a whole month I kept trying various relatively simple techniques so as to prove the existence of infinitely many such primes. It was natural to pick this topic for a research project and try to understand (*as a minimum*) *the theorem under consideration and its application to the specific problem. This turned out to be a difficult task – I had to study a lot of literature and spent numerous hours in the IMI library. I got invaluable help from Prof. Ivan Chipchakov, who explained to me in details the theorem and its applications.*

In the paper delivered at the student conference I showed a proof of a theorem of Ankeny and Rogers, attributed sometimes to E. Trost which implies as a particular case the generalization of the IMO problem. In addition I applied the result of Chebotarev and the approach of Rosen and Kraft so as to derive two completely different (in terms of philosophy) proofs of a theorem of Schinzel.

The work on my project was one of the happiest and most rewarding so far. I learned a very interesting and difficult theory and I realized what it means to struggle full-heartedly and systematically with solving of a complex mathematical problem. During this time I came across some interesting questions I wasn't able to solve.

This academic year I started a seminar with students (members of HSSI) with the thought of providing theoretical background for research projects. I felt that this would be an ideal opportunity for me to come back to my old project and pose some open problems to the students attending the seminar. To my great satisfaction, two students from Sofia High School of Mathematics (Ivan Penchev and Momchil Konstantinov) took the gauntlet down, acquired significant theoretical knowledge, and accomplished serious achievements. I am convinced that these topics are suitable for attracting young talents to mathematics since the applications of the theorem under discussion are related to the number theory problems whose formulation is easily accessible to high school students. The students could start with trying to prove some of the problems by means of elementary techniques, after which they would feel the strength and the beauty of Chebotarev's theorem. A natural work afterwards would be systematic study of Galois Theory and basic algebraic number theory which is necessary for getting deeper in the nature of the theorem itself and its applications. Thus, working in this field could contribute significantly to cultivating (early *enough) the necessary discipline and systematic work crucial for professional success in math research.*

Personally, I believe that thinking deeply about a particular math problem is the most important aspect of doing math research at school age. Most mathematically gifted high school students have very narrow knowledge of what real math is since their experience is based primarily on solving olympiad problems (usually designed to be tricky and lacking serious mathematical content). After the first year at the university many of these bright students switch to other fields since they realize that mathematics is not what they initially thought it to be. Undoubtfully, the olympiad movement has produced many outstanding mathematicians, but as time goes on, I am convinced that stimulating students to do research at school age will prove to be more valuable and efficient way of attracting bright young people to become professional mathematicians.

Example 3: Putting an old mathematics notion into a new context

Linda Brown Westrick (RSI 2003): The topic for this project was based on a notion appearing in a problem from the Tournament of Towns (2002 г.) in which a number derivative was defined for non-negative integers as follows:

(1) $p' = 1$ (if p is prime);
(2) $(mn)' = m'n + n'm$;
(3) $0' = 0$.

The name *derivative* is chosen thanks to the resemblance of the second rule with the Leibnitz rule for differentiation of a product of two functions. Thus in the case of 6 we get: $6' = (2.3)' = 2'.3 + 2.3' = 1.3 + 2.1 = 5$. Since $5' = 1$ the second derivative of 6 is 1, and the third one is 0.

The *number derivative* allows for various ideas from number theory to be explored in a new context. The interesting patterns emerging as a result of such a simple definition make the explorations of its properties worthwhile. It turns out that many open problems from number theory could be formulated by means of the number derivative. In her project Linda generalizes the notion of number derivative for rational numbers and writes a computer program so as to facilitate the exploration of its properties. She investigates the conditions under which the *differential equations* $x' = a$ and $x' = ax$ (where a and x are rational numebrs) have a solution. Furthermore, she examines the properties of the sequence of the consecutive number derivatives x, x', x'', ... and more specifically when this sequence is bounded for a given x. Her project won a grant in the Intel and Siemens competitions and was presented at the Youth session in the MASSEE Congress in 2003 [12, pp. 121–126].

Example 4: The endeavor for mathematical elegance

It is not rare when students and mentors argue which is more important – to achieve a new mathematical result or to experience the struggle for a beautiful proof of an already known result. We shall present a case when in a math context *the road is more important than the inn*:

Charles Tam (RSI 2005): *As a rising senior in RSI 2005, I made an attempt to solve a seemingly simple problem about partitions, brought to me by a friend as a mathematical curiosity. Suppose we have a Young diagram – a representation of a partition as a series of left-aligned blocks arranged in decreasing order, with the top row of the diagram corresponding to the largest piece of the partition. We call a cell in this diagram "nice"if the number of cells to its right in its row is exactly equal to, or one more* *than, the number of cells below it in its column. Now consider the set of all the partitions of n. It turns out that the multiset of the numbers of nice cells in each of these partitions is exactly the same as the multiset of the numbers of rows in each of the partitions! We can observe this by checking* $n = 5$ *(I myself was not convinced until I checked up to* $n = 11$*). It is immediately apparent that this problem has no chance to be practical in any sense; we are hard-pressed to even come up with a contrived mathematical problem that could make use of this fact. We therefore might expect the presence of some deeper mathematical truth. But its known proof yields little theoretical satisfaction; it requires interpreting the problem through the use of Hilbert schemes and complex manifolds* [13]. *I found it difficult (and still do) to believe that such a strange yet striking equality did not yield to any more elegant combinatorial proofs. As such, I endeavored to find a more elegant way to go about confirming the identity, hoping to link such a proof to the existing one and develop this hidden mathematical truth into a connection between combinatorics and topology. Ultimately, I failed to produce a complete proof for the problem, and the resulting paper was written as an exploration into strategies for securing a proof, with the execution left as an exercise to the readers. Years later, as an MIT student, I returned to the problem and found that the strategy of developing a recursive formula for each of the sets was almost what I wanted. But it remains, to me, unsolved, and I suspect it will continue to reside in the back of my mind until I am satisfied with its truth.*

Example 5: The strength of a metaphor when presenting shortly to a larger audience

One of the most difficult things concerning the oral presentation of the projects is how to convey deep mathematical ideas for 10 minutes to a larger audience of young scientists not necessarily mathematicians but eager to understand. Finding an appropriate metaphor is very helpful as the following extract of an oral presentation on Hyperbolic geometry project shows:

Bryant Mathews (RSI'97): *The idea came from a conversation I had with my mentor, Ioanid Rosu. When we say "geometry", we could mean almost anything. In order to specify exactly what we mean, we must define certain objects and how these objects are related. In geometry, the two basic objects are the point and the line. Two axioms which we want these to satisfy are as follows:*

1) *Two points lie on a unique line*;

2) *Between any two points on a line, there is another point which lies on the same line.*

We usually think of points as dots on a chalkboard, and lines as curves connecting these points. But what if we define these objects differently? Suppose we define a point to be an ant, and a line to be a colony of ants. A point "lies on" a certain line iff an ant is a member of a certain colony. Does this geometry satisfy the first axiom above? No, because two points need not lie on a line, i.e. two ants need not be in the same colony.

Let us try again. Suppose a point is an ant, and a line is a pair of ants. Now the first axiom is satisfied, because any two ants form a pair of ants. However, there is never a third ant on the same line, so the second axiom is not satisfied.

Thus, our two axioms rule out each of the ant geometries; however, many geometries remain. As we add more axioms, we rule out more and more geometries until we are left with only one. If we choose the axioms of Eucledean geometry, we are left with our common notions of point and line. If we choose a different set of axioms, we may be left, *for example, with the mysterious hyperbolic geometry, where rectangles do not exist.*

Twelve years later Bryant's students express their admiration for the clarity with which he presents very abstract mathematical ideas [14].

6. The art of being a math mentor. The term *mentor* comes from Odyssey, where Homer tells that Mentor became teacher of Telemachus and guided him in becoming a man during the absence of his father, Odysseus [15]. In the contemporary sense, most mentorships involving the gifted and talented are one-on-one, however high-ability youth could profit from multiple mentors, including peer mentors who can offer *emotional support, competition and comparison* at all stages of an individual's development [16].

The RSI mentors in mathematics have a slightly different status with respect to the mentors in other fields of the program. These are still students (undergraduates and graduates) who are themselves mentored how to be mentors by highly recognized professional mathematicians. After students have exposed their research interests and mathematical background in their essays the *mentor of mentors* discusses the research preferences of the RSI students with the mentors-to-be and matches them according to their respective research interests and background. Here is how Prof. Hartley Rogers of the MIT describes his involvement in RSI as a long term mentor and supervisor of mentors [17]: *I recruit each year a corps of RSI mentors from the rich and vibrant community of mathematics graduate students at MIT. Each mentor meets individually with each of his students for 1+ hours each day, and receives a stipend of $1250 per student from the MIT mathematics department. Before the start of RSI, I receive the admission files for the incoming RSI students in mathematics. Each mentor reads these files and submits his own ordered list of his top five preferred mentees. I examine these "ballots" and make a final matching of mentors and mentees. This system has been surprisingly successful. Beginning with the week before RSI starts, and continuing throughout the program, I convene a weekly general hour-long meeting of the mentors in which each mentor is given an opportunity to report on his own experience and progress, to ask questions or seek help, and to offer comment and advice to colleagues. The first two of these meetings are chiefly concerned with identifying good problems for the mentees to work on and with reports on negotiation and collaboration with mentees on choice of problem. Solving new mathematical problems is a chancy and unpredictable undertaking. In particular, the timing of success cannot be legislated in advance …But on the whole, the quality of the RSI students was so high, and the enthusiasm of the mentors was so great, that extraordinary results were achieved.*

Let us see what some of the math mentors think about their RSI mentoring with the hope to convey the flavor and the challenges such a work offers:

- **Mentor 1:** *I would describe my work at RSI as both pleasurable and difficult. The nice part is meeting talented and enthusiastic students who are eager to learn mathematics through their own investigations rather than one-way instruction. The difficult part is that mentoring students in a way that benefits them requires a great deal of thought and patience …*
- **Mentor 2:** *Designing a project depends greatly on the student's previous experience. …This makes the choice of a problem particularly crucial. The desire to have students work on unsolved problems makes this even harder. I would sacrifice the latter goal in favor of giving the students something they can get to grips with without too much hand-holding.*

Even from these short excerpts from the mentors' reports it is clear that there is great variety even among the gifted students and each project could lead to a surprise – *the problem is too easy for a 6-week project; the problem is too difficult for a 6-week project; the problem is interesting and suitable for your mentee but it turns out that it has been already solved a week before that*, etc. Thus the choice of a problem is crucial – it would be ideal if it is in the field of expertise of the mentor and of interest for the student but this is rarely the case. *When to make a compromise? Who should make it?* – all these questions are discussed among the mentors, their coordinator, and the tutors who in addition to helping with the presentations of the projects are responsible for the good psychological conditions of the students. The mentor is also expected to play various roles: teacher, expert, advisor, friend and role model. As difficult as these roles might be, to be a mentor is very rewarding – research shows that even short-term mentorships in gifted education can impact upon self-esteem, feelings of competence, and sense of worth [18].

In a national context, the teachers who are more experienced as mentors share their expertise with their colleagues at the special seminars in the frames of the RSS [Appendix, I].

7. Conclusion. The best reward for all those involved in nurturing and developing the research potential of young mathematicians while they are still at school is to see various forms of collaboration among students, teachers and researchers. In our recent experience within HSSI and RSI we are satisfied to see joint publications having emerged from students' research in the frames of these programs (cf. [12, pp. 135–140, 141–146, 151–157], [19]), and the effect of *passing the torch* to the next generation in new roles – mentors, lecturers, advisors. Some promising examples include HSSI projects mentored by Vesselin Dimitrov (Harvard), Alexander Lishkov (Princeton), Todor Bilarev (Bremen University), Konstantin Delchev (SU), Nikolay Dimitrov (SU), Antoni Rangachev (MIT).

Thus, the effect of enabling students to experience the research spirit of mathematics at school age is not only to attract young people to enrolment in this field but also to educate and prepare them in becoming good mentors and ambassadors for mathematics.

APPENDIX

I. SCIENTIFIC SUPPORT OF HSSI

Coordinators

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Boyanka Savova, National Math School
Artur Kirkorjan, SU
Dimitar Jetchev, Harvard University
Antoni Rangachev, MIT

II. MAIN TOPICS OF HSSI PROJECTS IN MATHEMATICS

ALGEBRA AND ANALYSIS

Algebraic and Geometric Inequalities
Convex functions
Recurrence relations
Polynomials
Functional equations
Elements of Galois theory

GEOMETRY

Geometry of triangle and quadrilateral
Metric properties of figures
Geometric extremum problems
Geometric transformations
Affine, analytic and projective geometry
Applications of complex numbers in geometry

NUMBER THEORY

Diophantine equations
Congruences
Arithmetic functions
Quadratic reciprocity
Miscellaneous – factorization in quadratic fields, Pell's equations, recurrence relations

COMBINATORICS

Graph Theory
Combinatorial Game Theory
Elements of Coding Theory
Combinatorial Geometry

Miscellaneous – pigeonhole principle, invariants and semi-invariants, colorings

III. SHORT-TERM RSS PROJECTS IN MATHEMATICS

RSS'05

- The Golab-Schinzel functional equation
- Simpson lines
- Malfatti problems
- Classical fractals
- Combinatorial games
- Catalan equation
- Diophantine equations
- Euler and Fermat theorems
- Exponential Diophantine equations
- The Fermat infinite descend method
- On the third problem from IMO'05
- Coverings of the plane

RSS'06

- Algebraic inequalities – quasilinearization
- Egyptian fractions
- Entire and fractional part of a number
- Orthogonal circles
- Inversion and orthotriangle
- Inversion and isogonal points
- A metric property of the equilateral triangle
- Sums of plane figures
- Geometric inequalities and complex numbers
- Sums of sets
- Combinatorial strategies
- Weight of coins
- Classical Diophantine equations
- The Frobenius coin problem

RSS'07

- Polynomial equations
- Complex numbers in geometry
- Compositions of geometric transformations
- Sangaku problems
- Vector cross product and oriented areas
- Algebraic methods for solving combinatorial problem

- Combinatorial problems for sequences
- Combinatorial geometry problems from BMO and IMO
- The Fermat infinite descend method
- Sets of sums with equal multiplicity
- Exponential Diophantine equations
- Zsigmondy's theorem
- Sums of squares

RSS'08

- Inequalities for sums of powers
- Diophantine equations for polynomials
- An application of Galois groups
- Insolvability of algebraic equations in radicals
- Constructive functional equations
- Remarkable lines trough remarkable points
- Geometric constructions
- Inscribed and subscribed triangles with given angles
- Correspondences between points and triangles generated by inversions
- The theorems of Pappus and Desargues
- Geometric inequalities from IMO
- Isogonal conjugation and the Fermat problem
- Ruler-and-compass constructions
- Binomial identities
- Colorings of the plane
- Codes and colorings
- Arithmetic dynamics

IV. BULGARIAN HSSI MATH STUDENTS AT RSI

Name	**Year**	**Title of the RSI project**
Kaloyan Slavov*	2001	*On Hurwitz equation and the related unicity conjecture*
Eleonora Encheva	2001	*A Generalization of Poncelot's theorem with application in cryptography*
Iva Rashkova	2002	*Graph Embeddings*
Vesselin Dimitrov*	2003	*Zero-sum problems in finite groups*
Antoni Rangachev*	2004	*On the solvability of p-adic diagonal equations*
Vladislav Petkov	2005	*The number of isomorphism classes of groups of order n and some related questions*
Todor Bilarev	2006	*Representations of integers as sums of square and triangular numbers*
Nikola Tchipev	2007	*On a linear Diophantine problem of Frobenius*
Galin Statev*	2008	*On Fermat-EulerDynamics*
Katrin Evtimova	2008	*Rational Cherednik algebras of rank 1 and 2*

*(five representative written presentation)

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ДА ВЛЕЗЕШ В СВЕТА НА МАТЕМАТИЧЕСКИТЕ ИЗСЛЕДВАНИЯ НА УЧЕНИЧЕСКА ВЪЗРАСТ

Олег Мушкаров, Антони Рангачев, Евгения Сендова

В доклада се разглеждат образователни програми за работа с ученици с изявени математически способности, като акцентът е върху разработването на изследователски проекти. Авторите споделят опита си и впечатленията си от такива програми в национален и международен контекст.